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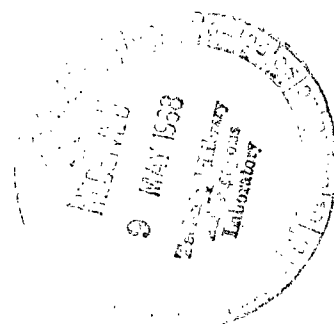
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# METHOD FOR DETERMINING NORMAL MODES AND FREQUENCIES OF A LAUNCH VEHICLE UTILIZING ITS COMPONENT NORMAL MODES

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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## ABSTRACT

The launch vehicle is idealized as a one-dimensional structure consisting of a main beam to which several flexible branches are attached. The normal modes of both the unrestrained main beam and the cantilevered branches are determined. An energy approach is then employed in which the displacement of the vehicle is expressed as the superposition of a finite number of free-free normal modes of the main beam and cantilever normal modes of the branches. Lagrange's equations are then used to derive the equations of motion in matrix form, and an iterative method of solution is included for completeness.

# METHOD FOR DETERMINING NORMAL MODES AND FREQUENCIES OF A LAUNCH VEHICLE UTILIZING ITS COMPONENT NORMAL MODES

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## SUMMARY

An analytical method is presented for determining the natural frequencies and lateral mode shapes of an idealized launch vehicle, which is envisioned as a one-dimensional structure consisting of a main beam to which several flexible branches are attached. Engines and sloshing masses are also included. The vehicle may be free or constrained by translational and rotational constraining springs.

An energy approach is employed in which the displacement shape of the system is expressed as the superposition of a finite number of free-free normal modes of the main beam and cantilever normal modes of the branches. The choice of these modes as assumed displacement functions provides a simple, straightforward method of incorporating branches, engines, and sloshing masses in the analysis; many terms of the energy expressions are eliminated and those terms that remain are easily evaluated. Lagrange's equations are then utilized to develop the equations of motion in matrix form, and an iterative method of solution is included for completeness.

## INTRODUCTION

In some instances, the launch vehicle can be appropriately idealized as a nonuniform Timoshenko beam for purposes of studying its lateral dynamic characteristics. In other cases, this model may not realistically represent the launch vehicle system. For instance, when large structural components are cantilevered within other portions of the vehicle, it becomes necessary to include them as flexible branches. An example of this is a spacecraft enclosed within a protective nose fairing, or an upper-stage engine support structure extending into the interstage adapter. The inclusion of these branches becomes especially significant when clearance problems are anticipated or when the dynamic loads within the branch itself must be evaluated. It may also be desirable to in-

clude the inertial effect of the engines, which are generally represented as constrained rigid bodies, and the effect of sloshing propellants.

The lateral normal modes of the launch vehicle system have been determined by a variety of methods. For example, in references 1 and 2, the main launch vehicle is analyzed as a nonuniform Timoshenko beam by the Myklestad method and the finite-element force method, respectively. In references 3 and 4, the vehicle is analyzed as a nonuniform Timoshenko branch beam system including engine effects by the Stodola method and the finite-element stiffness method, respectively.

In the present analysis, the effects of flexible branches, swiveling engines, sloshing masses, and constraining springs can be incorporated into the system in a straightforward manner. The normal modes of the unrestrained main beam and the normal modes of the cantilevered branches must first be determined. The displacement of the vehicle is then expressed as the superposition of rigid body displacement modes, normal modes of the free-free main beam, and cantilever normal modes of the branches. The energy expression is then formulated, and Lagrange's equations are applied to derive the differential equations of motion. The principle of utilizing the uncoupled normal modes of the system is not new; in reference 5, Scanlan proposes its application to airplanes, and Hurty (ref. 6), presents a general procedure for incorporating component modes in the finite-element displacement approach. The new feature in this analysis is the choice of this particular set of assumed modes to describe the normal modes of a launch vehicle system. This choice is logical when the nature of the connection of the branches in an actual launch vehicle is considered.

The present approach is attractive because the normal modes of the main beam or of the branches can be developed by the most suitable method and then easily incorporated in the system in terms of their modal characteristics. For example, a payload consisting of trusses and concentrated masses might be analyzed by the finite-element stiffness method and then coupled to the system by use of its cantilever normal modes, natural frequencies, and generalized masses. Also, describing a component in terms of its dynamic modes allows dynamic test data to be incorporated in the analysis.

## SYMBOLS

$A$	cross-sectional area of launch vehicle
$A_s$	effective shear area
$(A_s G)_i$	shear stiffness of $i^{\text{th}}$ beam
$B_i$	distance from attachment point of $i^{\text{th}}$ beam to its center of gravity ( $i > 0$ )

$B_0$	distance from center of gravity of system to center of gravity of main beam
$E$	Young's modulus
$EI_i$	bending stiffness of $i^{\text{th}}$ beam
$f$	total number of generalized coordinates used to describe motion of system
$f_i$	number of elastic generalized coordinates used to describe motion of $i^{\text{th}}$ beam
$G$	shear modulus
$I$	area moment of inertia of cross section
$J_{ei}$	mass moment of inertia of engine on $i^{\text{th}}$ beam about center of gravity of engine
$J_i$	mass moment of inertia of $i^{\text{th}}$ branch about its attachment point ( $i > 0$ )
$J_0$	mass moment of inertia of main beam about its center of gravity
$K_{ei}$	stiffness of rotational spring at gimbal point of engine on $i^{\text{th}}$ beam
$\bar{K}_r$	stiffness of $r^{\text{th}}$ sloshing spring
$K_{Tj}$	stiffness of $j^{\text{th}}$ translational constraining spring
$K_{\theta j}$	stiffness of $j^{\text{th}}$ rotational constraining spring
$k$	total number of branch beams (does not include main beam)
$L_r$	displacement of $r^{\text{th}}$ sloshing mass relative to equilibrium position
$\mathcal{L}_r$	displacement of $r^{\text{th}}$ sloshing mass relative to tank walls
$M_{in}$	modal moment at clamped end of $i^{\text{th}}$ branch for $n^{\text{th}}$ uncoupled mode ( $i = 1, 2, \dots, k; n = 1, 2, \dots, f_i$ )
$m_{ei}$	mass of engine on $i^{\text{th}}$ beam
$m_i$	mass of $i^{\text{th}}$ beam
$\bar{m}_r$	mass of $r^{\text{th}}$ sloshing mass
$Q_{in}$	modal shear at clamped end of $i^{\text{th}}$ branch for $n^{\text{th}}$ uncoupled mode ( $i = 1, 2, \dots, k; n = 1, 2, \dots, f_i$ )
$q_{in}$	generalized coordinate describing participation of $n^{\text{th}}$ uncoupled mode of $i^{\text{th}}$ beam in free vibration of system ( $i = 0, 1, \dots, k; n = 1, 2, \dots, f_i$ )
$q_{0R}$	generalized coordinate describing rigid body rotation
$q_{0T}$	generalized coordinate describing rigid body translation
$T$	kinetic energy of launch vehicle
$U$	potential energy of launch vehicle

$x$	longitudinal coordinate measured relative to center of gravity of system
$x_{ei}$	coordinate of gimbal point of engine on $i^{\text{th}}$ beam
$x_i$	coordinate of attachment point of $i^{\text{th}}$ beam ( $i > 0$ )
$\bar{x}_r$	coordinate of $r^{\text{th}}$ sloshing mass
$x_{Tj}$	coordinate of connection point of $j^{\text{th}}$ translational constraining spring
$x_{\theta j}$	coordinate of connection point of $j^{\text{th}}$ rotational constraining spring
$y_i(x)$	lateral displacement function of $i^{\text{th}}$ beam relative to equilibrium position
$\alpha_i$	angular rotation of engine on $i^{\text{th}}$ beam relative to bending slope at gimbal point
$\beta_i$	angular rotation of engine on $i^{\text{th}}$ beam relative to equilibrium position
$\epsilon_i$	distance from gimbal point of engine on $i^{\text{th}}$ beam to center of gravity of engine
$\theta_i(x)$	bending slope function of $i^{\text{th}}$ beam relative to equilibrium position
$\lambda_i$	displacement of center of gravity of swiveling engine on $i^{\text{th}}$ beam
$\mu_{in}$	generalized mass of $n^{\text{th}}$ uncoupled mode of $i^{\text{th}}$ beam ( $i = 0, 1, \dots, k$ ; $n = 1, \dots, f_i$ )
$\mu_s$	generalized mass of $s^{\text{th}}$ mode of vehicle
$\rho$	mass density
$\rho A_i$	mass per unit length of $i^{\text{th}}$ beam
$\rho I_i$	mass moment of inertia per unit length of $i^{\text{th}}$ beam
$\varphi_{in}(x)$	displacement function of $n^{\text{th}}$ uncoupled mode of $i^{\text{th}}$ beam ( $i = 0, 1, \dots, k$ ; $n = 1, 2, \dots, f_i$ )
$\varphi_{0R}(x)$	rigid body rotation displacement function
$\varphi_{0T}(x)$	rigid body translation displacement function
$\psi_{in}(x)$	bending slope function of $n^{\text{th}}$ uncoupled mode of $i^{\text{th}}$ beam ( $i = 0, 1, \dots, k$ ; $n = 1, 2, \dots, f_i$ )
$\psi_{0R}(x)$	rigid body rotation slope function
$\psi_{0T}(x)$	rigid body translation slope function
$\omega_{in}$	natural frequency of $n^{\text{th}}$ uncoupled mode of $i^{\text{th}}$ beam, rad/sec ( $i = 0, 1, \dots, k$ ; $n = 1, \dots, f_i$ )
$\omega_s$	natural frequency of $s^{\text{th}}$ mode of vehicle

Matrices:

[A] stiffness matrix

[B] inertia matrix  
 [I] unit matrix  
 {q} column matrix of generalized coordinates

Subscripts:

i beams of system ( $i = 0, 1, 2, \dots, k$ ) (main beam  $i = 0$ , branch beam  $i > 0$ )  
 n, p uncoupled modes of beam ( $n = 1, 2, \dots, f_i$ )

Superscripts:

$\cdot$  differentiation with respect to time  
 $'$  differentiation with respect to  $x$

## OUTLINE OF THEORETICAL DERIVATION

The displacement of the launch vehicle is expressed as the superposition of the rigid body translation and rotation of the system, and a finite number of free-free modes of the main beam and cantilever modes of the branches. The amplitude of each of these assumed modes is governed by a generalized coordinate. The engines, which can generally be considered as rigid in comparison with the supporting structure, are represented as rigid bodies pinned at the gimbal point and constrained by rotational springs, as shown in figure 1. If the dynamic effects of sloshing propellants are to be included, the usual practice (ref. 7) is to introduce a mechanical analogy (spring mass or pendulum) into the system. In this analysis, sloshing is represented as a spring mass. For the case of a constrained launch vehicle, translational and rotational constraining springs may be attached to the main beam. In all, the total number of generalized coordinates is equal to the number of assumed modes plus the number of engines and sloshing masses.

The kinetic and potential energies of the system  $T$  and  $U$ , respectively, are then expressed in terms of the assumed deflections and generalized coordinates. In the development of the energy expressions, both the main beam and the branches are treated as Timoshenko beams, but this is merely for ease of notation and does not affect the final equations. The fact that the assumed elastic modes are normal modes of specific components, and, hence, satisfy the orthogonality relations, eliminates many terms from the energy expressions. The remaining terms consist of physical and modal characteristics of the components; that is, mass, length, generalized mass, generalized forces, and natural frequencies. Applying Lagrange's equations yields the required differential equations of motion, the number of which is equal to the number of generalized



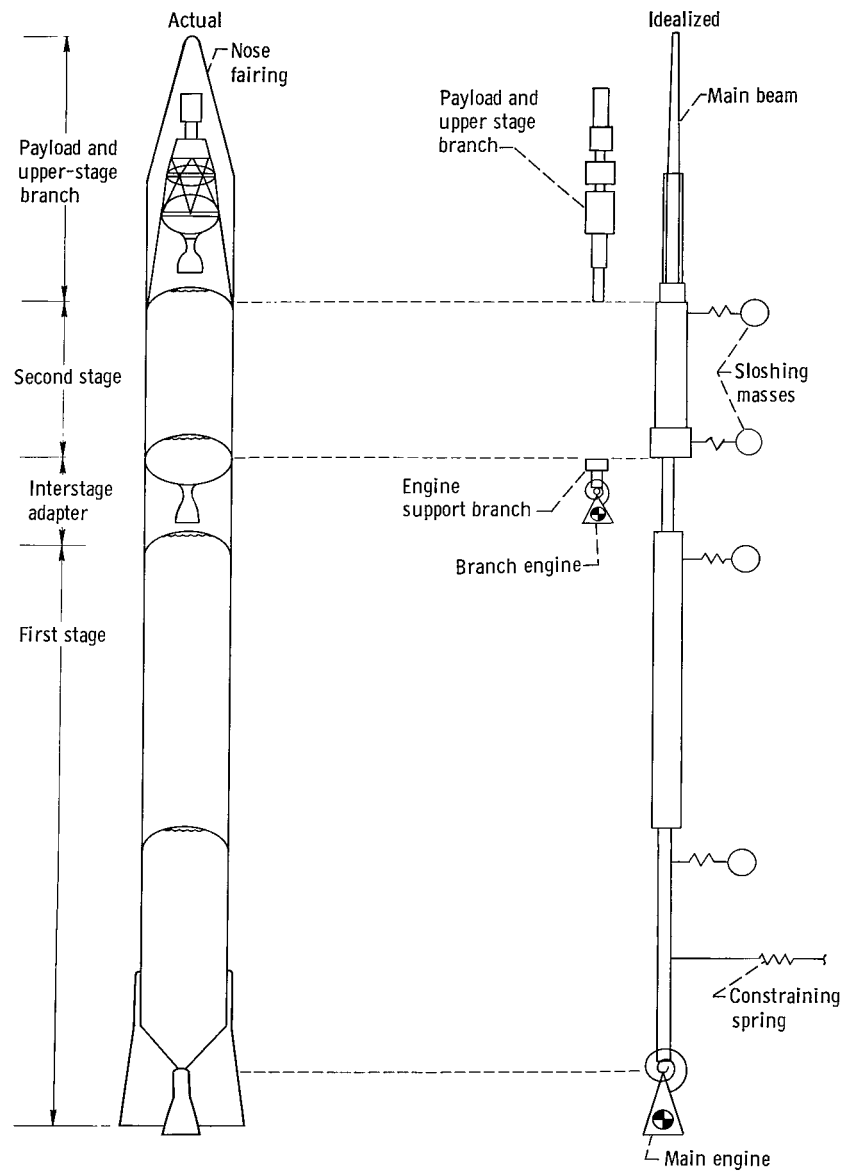


Figure 1. - Launch vehicle configuration.

coordinates. The equations in matrix form appear as

$$[B]\{\ddot{q}\} + [A]\{q\} = \{0\}$$

If harmonic motion is assumed, this equation is reduced to the eigenvalue equation

$$\left( \frac{1}{\omega^2} [I] - [A]^{-1}[B] \right) \{q\} = \{0\}$$

which is then solved for the natural frequencies and associated mode shapes by an iterative method.

### Coordinate System

As shown in figure 2, the launch vehicle configuration is described relative to a fixed x, y-coordinate system. The longitudinal coordinate x is measured relative to the center of gravity of the system.

The lateral displacement is assumed to be representable as the superposition of rigid body modes and uncoupled normal modes of the components. Following this ap-

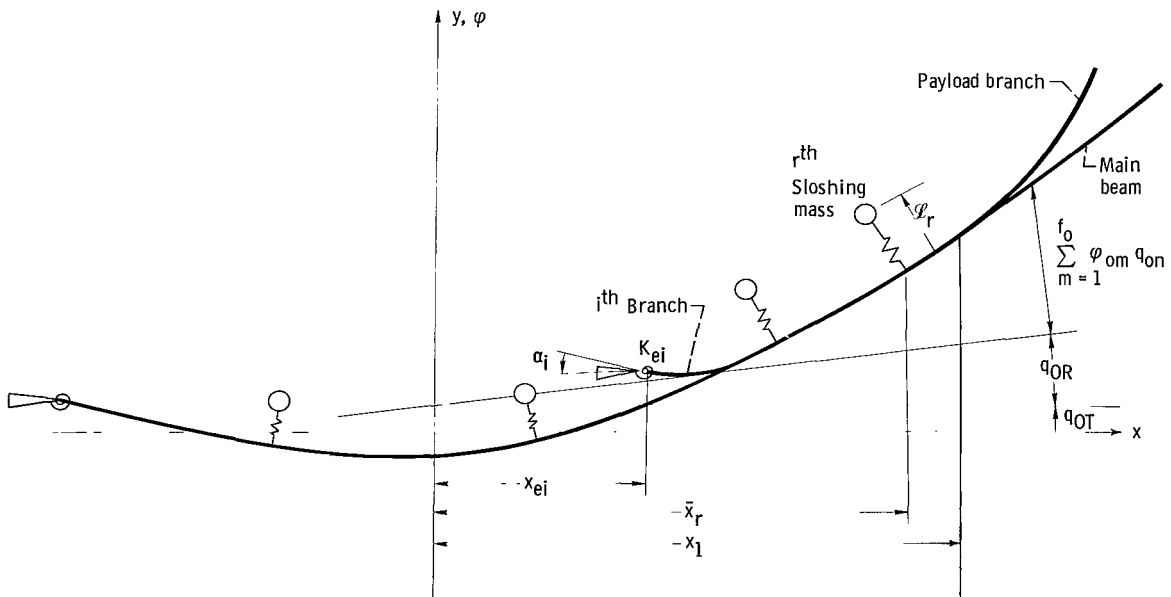


Figure 2. - Coordinate system.

proach, the deflection of the main beam  $y_0$  is expressed as the superposition of a rigid body translation mode  $\varphi_{0T} = 1$ ; a rigid body rotation mode  $\varphi_{0R} = x$ ; and the free-free normal modes of the main beam  $\varphi_{01}, \varphi_{02}, \dots, \varphi_{0f_0}$ ; that is,

$$\begin{aligned} y_0(x, t) &= \varphi_{0T}(x)q_{0T}(t) + \varphi_{0R}(x)q_{0R}(t) + \varphi_{01}(x)q_{01}(t) + \dots + \varphi_{0f_0}(x)q_{0f_0}(t) \\ &= \sum_{n=T, R, 1}^{f_0} \varphi_{0n}(x)q_{0n}(t) \end{aligned} \quad (1)$$

where the coefficients  $q_{0T}, q_{0R}, q_{01}, \dots, q_{0f_0}$  are generalized coordinates that describe the participation of the rigid body modes and free-free modes in the natural mode of vibration of the vehicle. Similarly, the bending slope of the main beam  $\theta_0$  is

$$\begin{aligned} \theta_0(x, t) &= \psi_{0T}(x)q_{0T}(t) + \psi_{0R}(x)q_{0R}(t) + \psi_{01}(x)q_{01}(t) + \dots + \psi_{0f_0}(x)q_{0f_0}(t) \\ &= \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x)q_{0n}(t) \end{aligned} \quad (2)$$

where

$\psi_{0T} = 0$	slope associated with translation mode $\varphi_{0T}$
$\psi_{0R} = 1$	slope associated with rotation mode $\varphi_{0R}$
$\psi_{01}, \psi_{02}, \dots, \psi_{0f_0}$	bending slope functions associated with free-free modes of main beam

The slope  $\psi_{0T}$ , associated with the translation mode, is zero but it is carried along in the analysis for simplicity in notation.

In like manner, the deflection of  $i^{\text{th}}$  branch  $y_i$  is expressed as the superposition of two rigid body modes associated with the translation and rotation of the branch as a whole, and the cantilever modes of the branch  $\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{if_i}$ ; that is,

$$y_i(x, t) = y_0(x_i, t) + \{x - x_i\} \theta_0(x_i, t) + \sum_{n=1}^{f_i} \varphi_{in}(x)q_{in}(t) \quad (i = 1, 2, \dots, k) \quad (3)$$

where  $y_0(x_i, t)$  is the displacement of the main beam at the attachment point  $x = x_i$  of the branch,  $\theta_0(x_i, t)$  is the bending slope of the main beam at the attachment point, and  $q_{i1}, q_{i2}, \dots, q_{if_i}$  are generalized coordinates describing the participation of the cantilever modes in the natural mode of vibration of the vehicle. When equations (1) and (2) are substituted into equation (3), the deflection is

$$y_i(x, t) = \sum_{n=T, R, 1}^{f_0} [\varphi_{0n}(x_i) + \{x - x_i\} \psi_{0n}(x_i)] q_{0n}(t) + \sum_{n=1}^{f_i} \varphi_{in}(x) q_{in}(t) \quad (i = 1, 2, \dots, k) \quad (4)$$

Similiarly, the bending slope of the  $i^{\text{th}}$  branch  $\theta_i$  is

$$\theta_i(x, t) = \theta_0(x_i, t) + \sum_{n=1}^{f_i} \psi_{in}(x) q_{in}(t) \quad (i = 1, 2, \dots, k) \quad (5)$$

where  $\psi_{i1}, \psi_{i2}, \dots, \psi_{if_i}$  are the bending slope functions associated with the cantilever modes of the branch. Substituting equation (2) into equation (5) gives the bending slope of the  $i^{\text{th}}$  branch

$$\theta_i(x, t) = \sum_{n=1}^{f_0} \psi_{0n}(x_i) q_{0n}(t) + \sum_{n=1}^{f_i} \psi_{in}(x) q_{in}(t) \quad (i = 1, 2, \dots, k) \quad (6)$$

After the deflection and bending slope of the main beam and branches are defined, the displacements of the swiveling engines and the sloshing masses can be defined. In the case of the engine, the displacement is expressed as the superposition of the motion of the clamped engine (the slope of the engine axis is equal to the bending slope at the gimbal point) and the vibratory motion relative to the clamped position. The displacement of the center of gravity of an engine is

$$\lambda_i(t) = y_i(x_{ei}, t) + \epsilon_i \theta_i(x_{ei}, t) + \epsilon_i \alpha_i(t) \quad (i = 0, 1, \dots, k) \quad (7)$$

where  $y_i(x_{ei}, t)$  is the displacement of the  $i^{\text{th}}$  beam at the gimbal point ( $x = x_{ei}$ ),  $\theta_i(x_{ei}, t)$

is the bending slope at the gimbal point,  $\epsilon_i$  is the distance from the gimbal point to the center of gravity of the engine, and  $\alpha_i$  is the generalized coordinate describing the angular rotation of the engine relative to the bending slope at the gimbal point.

The displacement of the center of gravity of a rigid engine can be expressed in terms of the generalized coordinates by substituting either equations (1) and (2) or equations (4) and (6) into equation (7), depending on whether the engine is attached to the main beam or a branch. Therefore, for an engine on the main beam,

$$\lambda_0(t) = \sum_{n=T, R, 1}^{f_0} [\varphi_{0n}(x_{e0}) + \epsilon_0 \psi_{0n}(x_{e0})] q_{0n}(t) + \epsilon_0 \alpha_0(t) \quad (8)$$

while for an engine on a branch,

$$\begin{aligned} \lambda_i(t) = & \sum_{n=T, R, 1}^{f_0} [\varphi_{0n}(x_i) + \{x_{ei} + \epsilon_i - x_i\} \psi_{0n}(x_i)] q_{0n}(t) \\ & + \sum_{n=1}^{f_i} [\varphi_{in}(x_{ei}) + \epsilon_i \psi_{in}(x_{ei})] q_{in}(t) + \epsilon_i \alpha_i(t) \quad (i = 1, 2, \dots, k) \end{aligned} \quad (9)$$

Following this same pattern, the angular rotation of the engine is

$$\beta_i(t) = \theta_i(x_{ei}, t) + \alpha_i(t) \quad (i = 0, 1, \dots, k) \quad (10)$$

Substituting equation (2) into equation (10) gives the angular rotation of an engine on the main beam as

$$\beta_0(t) = \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_{e0}) q_{0n}(t) + \alpha_0(t) \quad (11)$$

For the case of an engine on a branch, substitute equation (6) into equation (10)

$$\beta_i(t) = \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_i) q_{0n}(t) + \sum_{n=1}^{f_i} \psi_{in}(x_{ei}) q_{in}(t) + \alpha_i(t) \quad (i = 1, 2, \dots, k) \quad (12)$$

The displacement of the  $r^{\text{th}}$  sloshing mass  $L_r$  can be written in terms of  $y_0(\bar{x}_r, t)$ , the displacement of the main beam at the attachment point, and  $\mathcal{L}_r(t)$ , the generalized coordinate describing the displacement of the sloshing mass relative to the attachment point,

$$L_r(t) = y_0(\bar{x}_r, t) + \mathcal{L}_r(t) \quad (13)$$

Substituting equation (1) into equation (13) gives the displacement of the  $r^{\text{th}}$  sloshing mass

$$L_r(t) = \sum_{n=T, R, 1}^{f_0} \varphi_{0n}(\bar{x}_r) q_{0n}(t) + \mathcal{L}_r(t) \quad (14)$$

## Kinetic Energy

The use of Lagrange's equation requires the development of the kinetic energy expression in terms of the generalized velocities. For the idealized system of beams, masses, and engines, the kinetic energy is

$$T = \sum_{i=0}^k \left[ \frac{1}{2} \int_{l_i} \rho A_i \dot{y}_i^2 dx + \frac{1}{2} \int_{l_i} \rho I_i \dot{\theta}_i^2 dx + \frac{1}{2} m_{ei} \dot{\lambda}_i^2 + \frac{1}{2} J_{ei} \dot{\beta}_i^2 \right] + \frac{1}{2} \sum_r \bar{m}_r \dot{L}_r^2 \quad (15)$$

where the first term describes that energy associated with the lateral motion of the centerline; and  $\int_{l_i}$  refers to integration over the length of the  $i^{\text{th}}$  beam. The second term, the rotary inertia term, defines the energy associated with the rotation of an infinitesimal slice with the angular velocity  $\dot{\theta}$ . The following two terms result from the translation and rotation of the engine;  $\dot{\lambda}$  and  $\dot{\beta}$  are the lateral and angular velocities, respectively. The last term is the contribution of the sloshing mass.

Differentiating the displacement equations (eqs. (1), (2), (4), (6), (8), (9), (11), (12), and (14)) with respect to time and substituting the resulting equations into equation (15) yields a quadratic expression in the generalized velocities. Expanding this expression and interchanging the summation and integral signs provides the following

kinetic energy expression:

$$\begin{aligned}
T = & \frac{1}{2} \sum_{p=T, R, 1}^{f_0} \sum_{n=T, R, 1}^{f_0} \left( \int_{\ell_0} \rho A_0 \varphi_{0n} \varphi_{0p} dx + \int_{\ell_0} \rho I_0 \psi_{0n} \psi_{0p} dx \right) \dot{q}_{0n} \dot{q}_{0p} \\
& + \frac{1}{2} m_{e0} \left\{ \sum_{n=T, R, 1}^{f_0} \left[ \varphi_{0n}(x_{e0}) + \epsilon_0 \psi_{0n}(x_{e0}) \right] \dot{q}_{0n} + \epsilon_0 \dot{\alpha}_0 \right\}^2 \\
& + \frac{1}{2} J_{e0} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_{e0}) \dot{q}_{0n} + \dot{\alpha}_0 \right]^2 \\
& + \sum_{i=1}^k \left( \frac{1}{2} \sum_{p=T, R, 1}^{f_0} \sum_{n=T, R, 1}^{f_0} \left\{ \int_{\ell_i} \rho A_i \left[ \varphi_{0n}(x_i) + (x - x_i) \psi_{0n}(x_i) \right] \left[ \varphi_{0p}(x_i) + (x - x_i) \psi_{0p}(x_i) \right] dx \right. \right. \\
& \left. \left. + \int_{\ell_i} \rho I_i \psi_{0n}(x_i) \psi_{0p}(x_i) dx \right\} \dot{q}_{0n} \dot{q}_{0p} + \sum_{p=1}^{f_i} \sum_{n=T, R, 1}^{f_0} \left\{ \int_{\ell_i} \rho A_i \left[ \varphi_{0n}(x_i) + (x - x_i) \psi_{0n}(x_i) \right] \varphi_{ip} dx \right. \right. \\
& \left. \left. + \int_{\ell_i} \rho I_i \psi_{0n}(x_i) \psi_{ip} dx \right\} \dot{q}_{in} \dot{q}_{ip} + \frac{1}{2} \sum_{p=1}^{f_i} \sum_{n=1}^{f_i} \left( \int_{\ell_i} \rho A_i \varphi_{in} \varphi_{ip} dx + \int_{\ell_i} \rho I_i \psi_{in} \psi_{ip} dx \right) \dot{q}_{in} \dot{q}_{ip} \right. \\
& \left. + \frac{1}{2} m_{ei} \left\{ \sum_{n=T, R, 1}^{f_0} \left[ \varphi_{0n}(x_i) + (x_{ei} + \epsilon_i - x_i) \psi_{0n}(x_i) \right] \dot{q}_{0n} + \sum_{n=1}^{f_i} \left[ \varphi_{in}(x_{ei}) + \epsilon_i \psi_{in}(x_{ei}) \right] \dot{q}_{in} + \epsilon_i \dot{\alpha}_i \right\}^2 \right. \\
& \left. + \frac{1}{2} J_{ei} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_i) \dot{q}_{0n} + \sum_{n=1}^{f_i} \psi_{in}(x_{ei}) \dot{q}_{in} + \dot{\alpha}_i \right]^2 \right) \\
& + \sum_r \frac{1}{2} \bar{m}_r \sum_{n=T, R, 1}^{f_0} \left[ \varphi_{0n}(\bar{x}_r) \dot{q}_{0n} + \mathcal{L}_r \right]^2
\end{aligned} \tag{16}$$

The kinetic energy expression can be simplified by introducing the following conditions:

- (1) Conservation of linear and angular momentum for the uncoupled elastic free-free modes of the main beam; that is, preservation of translational and rotational equilibrium

$$\left. \begin{aligned} \int_{l_0} \rho A_0 \varphi_{0n} dx &= 0 \quad (n = 1, 2, \dots, f_0) \\ \int_{l_0} \rho A_0 x \varphi_{0n} dx + \int_{l_0} \rho I_0 \psi_{0n} dx &= 0 \quad (n = 1, 2, \dots, f_0) \end{aligned} \right\} \quad (17)$$

- (2) Inertial orthogonality and normalization condition for the uncoupled modes of the branch beams as shown in appendix A

$$\int_{l_i} \rho A_i \varphi_{in} \varphi_{ip} dx + \int_{l_i} \rho I_i \psi_{in} \psi_{ip} dx = \delta_{np} \mu_{in} \quad (i = 0, 1, 2, \dots, k) \quad (18)$$

(n, p = 1, 2, \dots, f\_i)

where  $\delta_{np}$  is the Kronecker delta defined such that

$$\delta_{np} = \begin{cases} 0 & (p \neq n) \\ 1 & (p = n) \end{cases} \quad (19)$$

and  $\mu_{in}$  is the generalized mass of the  $n^{\text{th}}$  uncoupled normal mode of the  $i^{\text{th}}$  beam.

Since the elastic modes of each branch are orthogonal relative to each other, a number of inertial coupling terms are eliminated from the kinetic energy expression. Those that remain are generally easily evaluated. To simplify the remaining terms further, the following notation is introduced:

$$m_0 B_0 = \int_{l_0} \rho A_0 x dx \quad (20a)$$

$$J_0 = \int_{l_0} \rho A_0 x^2 dx + \int_{l_0} \rho I_0 dx \quad (20b)$$

$$m_i B_i = \int_{l_i} \rho A_i (x - x_i) dx \quad (i = 1, 2, \dots, k) \quad (20c)$$



$$J_i = \int_{l_i} \rho A_i (x - x_i)^2 dx + \int_{l_i} \rho I_i dx \quad (i = 1, 2, \dots, k) \quad (20d)$$

$$-\frac{Q_{in}}{\omega_{in}^2} = \int_{l_i} \rho A_i \varphi_{in} dx \quad \begin{pmatrix} i = 1, 2, \dots, k \\ n = 1, 2, \dots, f_i \end{pmatrix} \quad (20e)$$

$$\frac{M_{in}}{\omega_{in}^2} = \int_{l_i} \rho A_i (x - x_i) \varphi_{in} dx + \int_{l_i} \rho I_i \psi_{in} dx \quad \begin{pmatrix} i = 1, 2, \dots, k \\ n = 1, 2, \dots, f_i \end{pmatrix} \quad (20f)$$

where

$m_0$  mass of main beam

$B_0$  distance from center of gravity of system to center of gravity of main beam

$J_0$  mass moment of inertia of main beam about center of gravity of system

$m_i$  mass of  $i^{\text{th}}$  branch

$B_i$  distance from center of gravity of system to attachment point of  $i^{\text{th}}$  branch

$J_i$  mass moment of inertia of  $i^{\text{th}}$  branch about attachment point

$Q_{in}$  modal shear at clamped end of  $i^{\text{th}}$  branch in  $n^{\text{th}}$  uncoupled cantilever mode (see fig. 3)

$\omega_{in}$  natural frequency of  $n^{\text{th}}$  uncoupled cantilever mode of  $i^{\text{th}}$  branch

$M_{in}$  modal moment at clamped end of  $i^{\text{th}}$  branch in  $n^{\text{th}}$  uncoupled cantilever mode (see fig. 3)

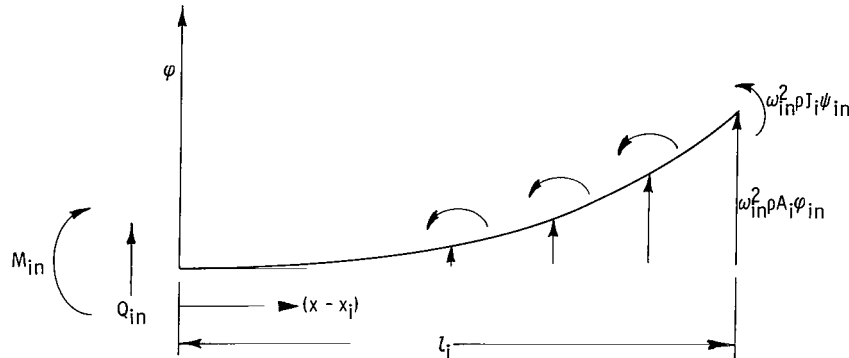


Figure 3. - Equilibrium of  $i^{\text{th}}$  cantilever branch in  $n^{\text{th}}$  mode.

The simplified kinetic energy expression now appears in terms of total masses, mass moments of inertia, natural frequencies, generalized masses, and generalized forces (modal shear and moment at base) of the components:

$$\begin{aligned}
T = & \frac{1}{2} \left( m_0 \dot{q}_{0T}^2 + 2 m_0 B_0 \dot{q}_{0T} \dot{q}_{0R} + J_0 \dot{q}_{0R}^2 + \sum_{n=1}^{f_0} \mu_{0n} \dot{q}_{0n}^2 \right) \\
& + \frac{1}{2} m_{e0} \left\{ \sum_{n=T, R, 1}^{f_0} \left[ \varphi_{0n}(x_{e0}) + \epsilon_0 \psi_{0n}(x_{e0}) \right] \dot{q}_{0n} + \epsilon_0 \dot{\alpha}_0 \right\}^2 \\
& + \frac{1}{2} J_{e0} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_{e0}) \dot{q}_{0n} + \dot{\alpha}_0 \right]^2 \\
& + \sum_{i=1}^k \left( \frac{1}{2} \sum_{p=T, R, 1}^{f_0} \sum_{n=T, R, 1}^{f_0} \left[ m_i \varphi_{0n}(x_i) \varphi_{0p}(x_i) + m_i B_i \psi_{0n}(x_i) \varphi_{0p}(x_i) \right. \right. \\
& \left. \left. + m_i B_i \varphi_{0n}(x_i) \psi_{0p}(x_i) + J_i \psi_{0n}(x_i) \psi_{0p}(x_i) \right] \dot{q}_{0n} \dot{q}_{0p} \right. \\
& \left. + \sum_{p=1}^{f_i} \sum_{n=T, R, 1}^{f_0} \left[ -\varphi_{0n}(x_i) \frac{Q_{ip}}{\omega_{ip}^2} + \psi_{0n}(x_i) \frac{M_{ip}}{\omega_{ip}^2} \right] \dot{q}_{0n} \dot{q}_{ip} + \frac{1}{2} \sum_{n=1}^{f_i} \mu_{in} \dot{q}_{in}^2 \right. \\
& \left. + \frac{1}{2} m_{ei} \left\{ \sum_{n=T, R, 1}^{f_0} \left[ \varphi_{0n}(x_i) + (x_{ei} + \epsilon_i - x_i) \psi_{0n}(x_i) \right] \dot{q}_{0n} \right. \right. \\
& \left. \left. + \sum_{n=1}^{f_i} \left[ \varphi_{in}(x_{ei}) + \epsilon_i \psi_{in}(x_{ei}) \right] \dot{q}_{in} + \epsilon_i \dot{\alpha}_i \right\}^2 \right. \\
& \left. + \frac{1}{2} J_{ei} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_i) \dot{q}_{0n} + \sum_{n=1}^{f_i} \psi_{in}(x_{ei}) \dot{q}_{in} + \dot{\alpha}_i \right]^2 \right) \\
& + \sum_{r=1} \frac{1}{2} \bar{m}_r \left[ \sum_{n=T, R, 1}^{f_0} \varphi_{0n}(\bar{x}_r) \dot{q}_{0n} + \dot{\mathcal{Z}}_r \right]^2
\end{aligned} \tag{21}$$

Thus far, the theoretical development has been restricted to beams; the kinetic energy and the orthogonality conditions are those of a beam. Suppose that the branch cannot be represented as a beam, but rather as a spacecraft composed of trusses and discrete masses. In this case, the final kinetic energy expression (eq. (21)) is still appropriate provided that the displacement can be expressed as the superposition of the translation and rotation of the branch as a whole, and of its cantilever normal modes. The mass properties and displacement functions in the kinetic energy expression (eq. (15)) then correspond to discrete points. The energy contribution involves coupling between the generalized coordinates describing the motion of the main beam and the branch. The orthogonality conditions again prevail to eliminate cross coupling terms between the generalized coordinates of the branch. The remaining coupling terms are simplified by expressing them in terms of the mass and mass moment of inertia of the branch, and the modal shears and moments developed at the base of the branch in the cantilever modes.

## Potential Energy

The potential energy associated with this system of beams, engines, constraining springs, and sloshing masses is

$$\begin{aligned}
 U = \sum_{i=0}^k & \left[ \frac{1}{2} \int_{l_i} EI_i \left( \frac{\partial}{\partial x} \theta_i \right)^2 dx + \frac{1}{2} \int_{l_i} (A_S G)_i \left( \theta_i - \frac{\partial}{\partial x} y_i \right)^2 dx + \frac{1}{2} K_{ei} \alpha_i^2 \right] \\
 & + \frac{1}{2} \sum_j K_{Tj} y_{0j}^2(x_{Tj}) + \frac{1}{2} \sum_m K_{\theta m} \theta_{0m}^2(x_{\theta m}) + \frac{1}{2} \sum_r \bar{K}_r \varphi_r^2 \quad (22)
 \end{aligned}$$

The first term describes that strain energy associated with beam bending, and the next term, the shear correction term, defines the energy associated with the shear slope  $\theta_i - (\partial y_i / \partial x)$ . The ensuing term includes the energy stored in the engine spring. The following two terms define the energy stored in the translational and rotational constraining springs, and the last term is the energy of the sloshing spring.

Differentiating equations (1), (2), (4), and (6), where the prime denotes differentiation with respect to  $x$ , yields

$$\left. \begin{aligned}
\frac{\partial y_0}{\partial x} &= \sum_{n=T, R, 1}^{f_0} \varphi'_{0n} q_{0n} \\
\frac{\partial y_i}{\partial x} &= \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_i) q_{0n} + \sum_{n=1}^{f_i} \varphi'_{in} q_{in} \\
\frac{\partial \theta_0}{\partial x} &= \sum_{n=T, R, 1}^{f_0} \psi'_{0n} q_{0n} \\
\frac{\partial \theta_i}{\partial x} &= \sum_{n=1}^{f_i} \psi'_{in} q_{in}
\end{aligned} \right\} \quad (23)$$

and substituting equation (23) into equation (22) yields the potential energy expression:

$$\begin{aligned}
& + K_{e0} \alpha_0^2 + \sum_{i=1}^k \left\{ \int_{l_i} EI_i \left( \sum_{n=1}^{f_i} \psi'_{in} q_{in} \right)^2 dx \right. \\
& + \int_{l_i} (A_S G)_i \left[ \sum_{n=1}^{f_i} (\psi_{in} - \varphi'_{in}) q_{in} \right]^2 dx + K_{ei} \alpha_i^2 \Big\} \\
& + \sum_j K_{Tj} \left[ \sum_{n=T, R, 1}^{f_0} \varphi_{0n}(x_{Tj}) q_{0n} \right]^2 \\
& + \sum_m K_{\theta m} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_{\theta m}) q_{0n} \right]^2 + \sum_r \bar{K}_r \varphi_r^2 \Big) \quad (24)
\end{aligned}$$

The potential energy may be simplified by introducing the following conditions:

(1) The rigid body modes and their derivatives must not contribute to the bending and shear strain energy of the beams:

$$\psi'_{0T} = \psi'_{0R} = (\psi_{0T} - \varphi'_{0T}) = (\psi_{0R} - \varphi'_{0R}) = 0 \quad (25)$$

(2) The stiffness orthogonality condition of the elastic uncoupled modes of the branch beams (appendix A) must be

$$\begin{aligned} \int_{l_i} EI_i \psi'_{in} \psi'_{ip} dx + \int_{l_i} (A_s G)_i (\psi_{in} - \varphi'_{in}) (\psi_{ip} - \varphi'_{ip}) dx \\ = \delta_{np} \mu_{in} \omega_{in}^2 \quad (i = 0, 1, 2, \dots, k) \\ (n, p = 1, 2, \dots, f_i) \end{aligned} \quad (26)$$

where  $\delta_{np}$  is the Kronecker Delta defined in equation (19), and  $\mu_{in} \omega_{in}^2$  is referred to as the generalized stiffness of the  $n^{\text{th}}$  uncoupled mode of the  $i^{\text{th}}$  beam.

Again, the orthogonality condition, characteristic of the normal modes, proves invaluable in eliminating cross coupling terms. Its merits are obvious in the following potential energy expression, which results from substituting equations (25) and (26) into equation (24).

$$\begin{aligned} U = \sum_{i=0}^k \left( \sum_{n=1}^{f_i} \frac{1}{2} \mu_{in} \omega_{in}^2 q_{in}^2 + \frac{1}{2} K_{ei} \alpha_i^2 \right) + \frac{1}{2} \sum_j K_{Tj} \left[ \sum_{n=T, R, 1}^{f_0} \varphi_{0n}(x_{Tj}) q_{0n} \right]^2 \\ + \frac{1}{2} \sum_m K_{\theta m} \left[ \sum_{n=T, R, 1}^{f_0} \psi_{0n}(x_{\theta m}) q_{0n} \right]^2 + \frac{1}{2} \sum_r \bar{K}_r \varphi_r^2 \end{aligned} \quad (27)$$

Examination of the potential energy expression reveals that the terms pertaining to the branch beams, swiveling engines, and sloshing masses have reduced to the canonical form  $\sum_i a_i q_i^2$  of the quadratic, while those terms associated with the constraining springs remain  $\left( \sum_i a_i q_i \right)^2$ . When this function is substituted into Lagrange's equation, the quadratic terms will produce cross products of generalized coordinates (static cou-

pling terms) in the equations of motion. Equation (27) reveals that static coupling is introduced only by the constraining springs, while the addition of terms for branch beams, swiveling engines, and sloshing masses does not result in static coupling.

## Equations of Motion

The kinetic energy and potential energy equations ((21) and (27)) have been evaluated in terms of the generalized coordinates. It is now possible to introduce Lagrange's equation

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) + \frac{\partial U}{\partial q_i} = 0 \quad (28)$$

The application of this equation supplies the "f" equations of motion, each of which corresponds to a specific generalized coordinate. Because of the formidable appearance of the equations of motion, it is natural to bypass the present method of notation by the introduction of matrix notation. In matrix form, the "f" equations of motion appear as follows:

$$[B]\{\ddot{q}\} + [A]\{q\} = \{0\} \quad (29)$$

where

[B] inertia matrix describing coupling of various masses of system

{q} column matrix of generalized coordinates

[A] stiffness matrix defining coupling effect of various stiffnesses of system

Again, the complicated nature of the matrices, especially the inertia matrix, warrants special consideration in their presentation. Each matrix is decomposed into several elementary matrices, and the stiffness and inertia matrices are presented as sums of these matrices. The elementary matrices pertain to specific components, such as branches, engines, sloshing masses, and constraining springs. In this manner, various components may be added to or removed from the system at will, so that

$$[A] = \sum_{i=0}^k ([A_i] + [A_{ei}]) + \sum_r [A_{sr}] + [A_T] + [A_\theta] \quad (30)$$

$$[B] = \sum_{i=0}^k ([B_i] + [B_{ei}]) + \sum_r [B_{sr}] \quad (31)$$

where

- $[A_i], [B_i]$  stiffness and inertia matrices, respectively, for  $i^{\text{th}}$  branch  
( $i = 0, 1, 2 \dots$ )
- $[A_{ei}], [B_{ei}]$  stiffness and inertia matrices, respectively, for engine on  $i^{\text{th}}$  branch
- $[A_{sr}], [B_{sr}]$  stiffness and inertia matrices, respectively, for  $r^{\text{th}}$  sloshing mass
- $[A_T], [A_\theta]$  stiffness matrices for translational and rotational constraining springs

The formation of these matrices can best be shown by an example. Consider a system that consists of a main beam with an engine, two branch beams with engines, a sloshing mass, translational constraining springs, and rotational constraining springs. The displacement of the system is described in terms of the following generalized coordinates:

- $q_{0T}, q_{0R}$  rigid body coordinates
- $q_{01}, q_{02}, q_{03}$  three elastic coordinates of main beam
- $\alpha_0$  rotation of engine on main beam
- $q_{11}, q_{12}, q_{13}$  three elastic coordinates of first branch beam
- $\alpha_1$  rotation of engine on first branch beam
- $q_{21}, q_{22}$  two elastic coordinates of second branch beam
- $\alpha_2$  rotation of engine on second branch beam
- 1 displacement of the sloshing mass

The position of the generalized coordinates in the column matrix is the same as the list just given; for example, the transpose of the column matrix appears as

$$\{q\}^T = [q_{0T} \ q_{0R} \ q_{01} \ q_{02} \ q_{03} \ \alpha_0 \ q_{11} \ q_{12} \ q_{13} \ \alpha_1 \ q_{21} \ q_{22} \ \alpha_2 \ \mathcal{L}_1]$$

The general procedure for the placement of the generalized coordinates in the column matrix is the same as stated previously; rigid body coordinates ( $q_{0T}, q_{0R}$ ), main beam and engine coordinates ( $q_{01}, q_{02}, \dots, q_{0f_0}, \alpha_0$ ), branch beam and engine coordinates ( $q_{11}, q_{12}, \dots, q_{1f_1}, \alpha_1, q_{21}, q_{22}, \dots, q_{2f_2}, \alpha_2, \dots, q_{k1}, q_{k2}, \dots, q_{kf_k}, \alpha_k$ ), and the sloshing coordinates ( $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_r$ ). The elementary stiffness and mass matrices are now presented.

Main beam. - For the main beam, the stiffness and mass matrices are

$$\begin{aligned}
 [A_0] &= \begin{bmatrix} q_{0T} & q_{0R} & q_{01} & q_{02} & q_{03} & \alpha_0 & \cdot & \cdot & \cdot & \mathcal{L}_1 \\ 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & \mu_{01}\omega_{01}^2 & & & & & & & \\ & & & \mu_{02}\omega_{02}^2 & & & & & & \\ & & & & \mu_{03}\omega_{03}^2 & & & & & \\ & & & & & 0 & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 0 \end{bmatrix} \\
 [B_0] &= \begin{bmatrix} m_0 & m_0 B_0 & & & & & & & & \\ m_0 B_0 & J_0 & & & & & & & & \\ & & \mu_{01} & & & & & & & \\ & & & \mu_{02} & & & & & & \\ & & & & \mu_{03} & & & & & \\ & & & & & 0 & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 0 \end{bmatrix}
 \end{aligned}$$



Engine on main beam. - With an engine on the main beam, the stiffness and mass matrices are

$$\begin{aligned}
 & \boxed{q_{0T} \quad q_{0R} \quad q_{01} \quad q_{02} \quad q_{03} \quad \alpha_0 \quad q_{11} \quad \cdot \quad \cdot \quad \cdot \quad \mathcal{L}_1} \\
 [A_{e0}] &= \begin{bmatrix} 0 & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 0 & & & & & & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & 0 & & & & & \\ & & & & & & k_{e0} & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & \cdot & & \\ & & & & & & & & & \cdot & \\ & & & & & & & & & & \cdot \\ & & & & & & & & & & & 0 \end{bmatrix} \\
 [B_{e0}] &= m_{e0} \left\{ \begin{array}{c} 1 \\ x_{e0} + \epsilon_0 \\ \varphi_{01}(x_{e0}) + \epsilon_0 \psi_{01}(x_{e0}) \\ \varphi_{02}(x_{e0}) + \epsilon_0 \psi_{02}(x_{e0}) \\ \varphi_{03}(x_{e0}) + \epsilon_0 \psi_{03}(x_{e0}) \\ \epsilon_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \quad [\text{Transpose of column matrix}] \quad \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{array}} \\
 & + J_{e0} \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_{e0}) \\ \psi_{02}(x_{e0}) \\ \psi_{03}(x_{e0}) \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \quad [\text{Transpose of column matrix}] \quad \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{array}}
 \end{aligned}$$

Main beam. - For the main beam, the stiffness and mass matrices are

$$\begin{aligned}
 & \begin{bmatrix} q_{0T} & q_{0R} & q_{01} & q_{02} & q_{03} & \alpha_0 & \cdot & \cdot & \cdot & \mathcal{L}_1 \end{bmatrix} \\
 [A_0] = & \begin{bmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & \mu_{01}\omega_{01}^2 & & & & & & & \\ & & & \mu_{02}\omega_{02}^2 & & & & & & \\ & & & & \mu_{03}\omega_{03}^2 & & & & & \\ & & & & & 0 & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 0 \end{bmatrix} \\
 [B_0] = & \begin{bmatrix} m_0 & m_0 B_0 & & & & & & & & \\ m_0 B_0 & J_0 & & & & & & & & \\ & & \mu_{01} & & & & & & & \\ & & & \mu_{02} & & & & & & \\ & & & & \mu_{03} & & & & & \\ & & & & & 0 & & & & \\ & & & & & & \cdot & & & \\ & & & & & & & \cdot & & \\ & & & & & & & & \cdot & \\ & & & & & & & & & 0 \end{bmatrix}
 \end{aligned}$$

Engine on main beam. - With an engine on the main beam, the stiffness and mass matrices are

$$\begin{aligned}
 & \begin{bmatrix} q_{0T} & q_{0R} & q_{01} & q_{02} & q_{03} & \alpha_0 & q_{11} & \cdot & \cdot & \cdot & \mathcal{L}_1 \end{bmatrix} \\
 [A_{e0}] = & \begin{bmatrix} 0 & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 0 & & & & & & & & \\ & & & 0 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & k_{e0} & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & \cdot & & & \\ & & & & & & & & \cdot & & \\ & & & & & & & & & \cdot & \\ & & & & & & & & & & 0 \end{bmatrix} \\
 [B_{e0}] = m_{e0} & \left\{ \begin{array}{c} 1 \\ x_{e0} + \epsilon_0 \\ \varphi_{01}(x_{e0}) + \epsilon_0 \psi_{01}(x_{e0}) \\ \varphi_{02}(x_{e0}) + \epsilon_0 \psi_{02}(x_{e0}) \\ \varphi_{03}(x_{e0}) + \epsilon_0 \psi_{03}(x_{e0}) \\ \epsilon_0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \begin{array}{l} \text{[Transpose of column matrix]} \\ \begin{bmatrix} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{bmatrix} \end{array} \\
 + J_{e0} & \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_{e0}) \\ \psi_{02}(x_{e0}) \\ \psi_{03}(x_{e0}) \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \begin{array}{l} \text{[Transpose of column matrix]} \\ \begin{bmatrix} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{bmatrix} \end{array}
 \end{aligned}$$

First branch beam. - For the first branch beam, the stiffness and mass matrices are

$$\begin{aligned}
 [A_1] &= \begin{bmatrix} q_{0T} & q_{0R} & q_{01} & q_{02} & q_{03} & \alpha_0 & q_{11} & q_{12} & q_{13} & \alpha_1 & \cdot & \cdot & \cdot & \mathcal{L}_1 \\ 0 & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & \\ & & 0 & & & & & & & & & & & \\ & & & 0 & & & & & & & & & & \\ & & & & 0 & & & & & & & & & \\ & & & & & 0 & & & & & & & & \\ & & & & & & \mu_{11}\omega_{11}^2 & & & & & & & \\ & & & & & & & \mu_{12}\omega_{12}^2 & & & & & & \\ & & & & & & & & \mu_{13}\omega_{13}^2 & & & & & \\ & & & & & & & & & 0 & & & & \\ & & & & & & & & & & \cdot & & & \\ & & & & & & & & & & & \cdot & & \\ & & & & & & & & & & & & \cdot & \\ & & & & & & & & & & & & & 0 \end{bmatrix} \\
 [B_1] &= \begin{bmatrix} 0 & & & & & & & & & & & & & \\ & 0 & & & & & & & & & & & & \\ & & 0 & & & & & & & & & & & \\ & & & 0 & & & & & & & & & & \\ & & & & 0 & & & & & & & & & \\ & & & & & 0 & & & & & & & & \\ & & & & & & 0 & & & & & & & \\ & & & & & & & \mu_{11} & & & & & & \\ & & & & & & & & \mu_{12} & & & & & \\ & & & & & & & & & \mu_{13} & & & & \\ & & & & & & & & & & 0 & & & \\ & & & & & & & & & & & \cdot & & \\ & & & & & & & & & & & & \cdot & \\ & & & & & & & & & & & & & \cdot \\ & & & & & & & & & & & & & & 0 \end{bmatrix} \\
 + \left\{ \begin{array}{c} 1 \\ x_1 \\ \varphi_{01}(x_1) \\ \varphi_{02}(x_1) \\ \varphi_{03}(x_1) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} & \begin{bmatrix} 0 & \cdot & \cdot & \cdot & \cdot & 0 & \frac{-Q_{11}}{\omega_{11}^2} & \frac{-Q_{12}}{\omega_{12}^2} & \frac{-Q_{13}}{\omega_{13}^2} & 0 & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}
 \end{aligned}$$

$$\begin{array}{c}
\boxed{q_{0T} \quad q_{0R} \quad q_{01} \quad q_{02} \quad q_{03} \quad \alpha_0 \quad . \quad . \quad . \quad \mathcal{L}_1} \\
+ m_1 \left\{ \begin{array}{c} 1 \\ x_1 \\ \varphi_{01}(x_1) \\ \varphi_{02}(x_1) \\ \varphi_{03}(x_1) \\ 0 \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccc} 1 & x_1 & \varphi_{01}(x_1) & \varphi_{02}(x_1) & \varphi_{03}(x_1) & 0 & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}} \\
+ m_1 B_1 \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_1) \\ \psi_{02}(x_1) \\ \psi_{03}(x_1) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccc} 1 & x_1 & \varphi_{01}(x_1) & \varphi_{02}(x_1) & \varphi_{03}(x_1) & 0 & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}} \\
+ m_1 B_1 \left\{ \begin{array}{c} 1 \\ x_1 \\ \varphi_{01}(x_1) \\ \varphi_{02}(x_1) \\ \varphi_{03}(x_1) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccc} 0 & 1 & \psi_{01}(x_1) & \psi_{02}(x_1) & \psi_{03}(x_1) & 0 & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}} \\
+ J_1 \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_1) \\ \psi_{02}(x_1) \\ \psi_{03}(x_1) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccc} 0 & 1 & \psi_{01}(x_1) & \psi_{02}(x_1) & \psi_{03}(x_1) & 0 & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}}
\end{array}$$

$$\begin{array}{c}
\boxed{q_{0T} \quad q_{0R} \quad q_{01} \quad q_{02} \quad q_{03} \quad \alpha_0 \quad q_{11} \quad q_{12} \quad q_{13} \quad . \quad . \quad . \quad \mathcal{L}_1} \\
+ \left\{ \begin{array}{c} 0 \\ . \\ . \\ . \\ \frac{-Q_{11}}{\omega_{11}^2} \\ \frac{-Q_{12}}{\omega_{12}^2} \\ \frac{-Q_{13}}{\omega_{13}^2} \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccccccc} 1 & x_1 & \varphi_{01}(x_1) & \varphi_{02}(x_1) & \varphi_{03}(x_1) & 0 & . & . & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ . \\ . \\ . \\ q_{11} \\ q_{12} \\ q_{13} \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}} \\
+ \left\{ \begin{array}{c} 0 \\ . \\ . \\ . \\ \frac{M_{11}}{\omega_{11}^2} \\ \frac{M_{12}}{\omega_{12}^2} \\ \frac{M_{13}}{\omega_{13}^2} \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccccccc} 0 & 1 & \psi_{01}(x_1) & \psi_{02}(x_1) & \psi_{03}(x_1) & 0 & . & . & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ . \\ . \\ . \\ q_{11} \\ q_{12} \\ q_{13} \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}} \\
+ \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_1) \\ \psi_{02}(x_1) \\ \psi_{03}(x_1) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\} \left[ \begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & \frac{M_{11}}{\omega_{11}^2} & \frac{M_{12}}{\omega_{12}^2} & \frac{M_{13}}{\omega_{13}^2} & . & . & . & 0 \end{array} \right] \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1 \end{array}}
\end{array}$$

Engine on first branch beam. - For an engine on the branch beam, the stiffness and mass matrices are

$$\begin{array}{c}
 \boxed{q_{0T} \quad q_{0R} \quad q_{01} \quad q_{02} \quad q_{03} \quad \alpha_0 \quad q_{11} \quad q_{12} \quad q_{13} \quad \alpha_1 \quad q_{21} \quad q_{22} \quad \alpha_2 \quad \mathcal{L}_1} \\
 \\
 [A_{e1}] = \begin{bmatrix}
 0 & & & & & & & & & & & & & \\
 & 0 & & & & & & & & & & & & \\
 & & 0 & & & & & & & & & & & \\
 & & & 0 & & & & & & & & & & \\
 & & & & 0 & & & & & & & & & \\
 & & & & & 0 & & & & & & & & \\
 & & & & & & 0 & & & & & & & \\
 & & & & & & & 0 & & & & & & \\
 & & & & & & & & 0 & & & & & \\
 & & & & & & & & & k_{e1} & & & & \\
 & & & & & & & & & & 0 & & & \\
 & & & & & & & & & & & 0 & & \\
 & & & & & & & & & & & & 0 & \\
 & & & & & & & & & & & & & 0
 \end{bmatrix} \\
 \\
 [B_{e1}] = m_{e1} \left\{ \begin{array}{c}
 1 \\
 x_{e1} + \epsilon_1 \\
 \varphi_{01}(x_1) + (x_{e1} + \epsilon_1 - x_1)\psi_{01}(x_1) \\
 \varphi_{02}(x_1) + (x_{e1} + \epsilon_1 - x_1)\psi_{02}(x_1) \\
 \varphi_{03}(x_1) + (x_{e1} + \epsilon_1 - x_1)\psi_{03}(x_1) \\
 0 \\
 \varphi_{11}(x_{e1}) + \epsilon_1\psi_{11}(x_{e1}) \\
 \varphi_{12}(x_{e1}) + \epsilon_1\psi_{12}(x_{e1}) \\
 \varphi_{13}(x_{e1}) + \epsilon_1\psi_{13}(x_{e1}) \\
 \epsilon_1 \\
 \cdot \\
 \cdot \\
 \cdot \\
 0
 \end{array} \right\} \quad \begin{array}{l} \text{[Transpose of column matrix]} \end{array} \quad \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ q_{12} \\ q_{13} \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{array}} \\
 \\
 + J_{e1} \left\{ \begin{array}{c}
 0 \\
 1 \\
 \psi_{01}(x_1) \\
 \psi_{02}(x_1) \\
 \psi_{03}(x_1) \\
 0 \\
 \psi_{11}(x_{e1}) \\
 \psi_{12}(x_{e1}) \\
 \psi_{13}(x_{e1}) \\
 1 \\
 \cdot \\
 \cdot \\
 \cdot \\
 0
 \end{array} \right\} \quad \begin{array}{l} \text{[Transpose of column matrix]} \end{array} \quad \boxed{\begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ q_{11} \\ q_{12} \\ q_{13} \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{array}}
 \end{array}$$

Second branch beam. - For the second branch, the stiffness and mass matrices are

$$\begin{aligned}
 & \begin{matrix} q_{0T} & \cdot & \cdot & \cdot & \alpha_1 & q_{21} & q_{22} & \alpha_2 & \mathcal{L}_1 \end{matrix} \\
 [A_2] = & \begin{bmatrix} 0 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & 0 & & & & \\ & & & & & \mu_{21}\omega_{21}^2 & & & \\ & & & & & & \mu_{22}\omega_{22}^2 & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix} \\
 [B_2] = & \begin{bmatrix} 0 & & & & & & & & \\ & \cdot & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & 0 & & & & \\ & & & & & \mu_{21} & & & \\ & & & & & & \mu_{22} & & \\ & & & & & & & 0 & \\ & & & & & & & & 0 \end{bmatrix} \\
 + \begin{Bmatrix} 1 \\ x_2 \\ \varphi_{01}(x_2) \\ \varphi_{02}(x_2) \\ \varphi_{03}(x_2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} & \begin{bmatrix} 0 & \cdot & \cdot & \cdot & 0 & \frac{-Q_{21}}{\omega_{21}^2} & \frac{-Q_{22}}{\omega_{22}^2} & 0 & 0 \end{bmatrix} \begin{matrix} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{matrix}
 \end{aligned}$$



$$\begin{array}{c}
\begin{array}{c}
\boxed{q_{0T} \quad q_{0R} \quad q_{01} \quad q_{02} \quad q_{03} \quad \alpha_0 \quad . \quad . \quad . \quad \mathcal{L}_1} \\
\left[ \begin{array}{cccccc} 1 & x_2 & \varphi_{01}(x_2) & \varphi_{02}(x_2) & \varphi_{03}(x_2) & 0 & . & . & . & 0 \end{array} \right]
\end{array}
\end{array}
\left\{ \begin{array}{c} 1 \\ x_2 \\ \varphi_{01}(x_2) \\ \varphi_{02}(x_2) \\ \varphi_{03}(x_2) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\}
\begin{array}{c}
\boxed{q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1}
\end{array}$$

$$+ m_2 \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_2) \\ \psi_{02}(x_2) \\ \psi_{03}(x_2) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\}
\left[ \begin{array}{cccccc} 1 & x_2 & \varphi_{01}(x_2) & \varphi_{02}(x_2) & \varphi_{03}(x_2) & 0 & . & . & . & 0 \end{array} \right]
\begin{array}{c}
\boxed{q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1}
\end{array}$$

$$+ m_2 B_2 \left\{ \begin{array}{c} 1 \\ x_2 \\ \varphi_{01}(x_2) \\ \varphi_{02}(x_2) \\ \varphi_{03}(x_2) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\}
\left[ \begin{array}{cccccc} 0 & 1 & \psi_{01}(x_2) & \psi_{02}(x_2) & \psi_{03}(x_2) & 0 & . & . & . & 0 \end{array} \right]
\begin{array}{c}
\boxed{q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1}
\end{array}$$

$$+ J_2 \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_2) \\ \psi_{02}(x_2) \\ \psi_{03}(x_2) \\ 0 \\ . \\ . \\ . \\ 0 \end{array} \right\}
\left[ \begin{array}{cccccc} 0 & 1 & \psi_{01}(x_2) & \psi_{02}(x_2) & \psi_{03}(x_2) & 0 & . & . & . & 0 \end{array} \right]
\begin{array}{c}
\boxed{q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ . \\ . \\ . \\ \mathcal{L}_1}
\end{array}$$

$$\begin{array}{c}
\left[ \begin{array}{cccccc} q_{0T} & q_{0R} & q_{01} & q_{02} & q_{03} & \alpha_0 & \cdot & \cdot & \cdot & \mathcal{L}_1 \end{array} \right] \\
+ \left\{ \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{-Q_{21}}{\omega_{21}^2} \\ \frac{-Q_{22}}{\omega_{22}^2} \\ 0 \\ 0 \end{array} \right\} \left[ \begin{array}{cccccc} 1 & x_2 & \varphi_{01}(x_2) & \varphi_{02}(x_2) & \varphi_{03}(x_2) & 0 & \cdot & \cdot & \cdot & 0 \end{array} \right] \left[ \begin{array}{c} q_{0T} \\ \cdot \\ \cdot \\ \cdot \\ q_{21} \\ q_{22} \\ \alpha_2 \\ \mathcal{L}_1 \end{array} \right]
\end{array}$$

$$\begin{array}{c}
\left\{ \begin{array}{c} 0 \\ \cdot \\ \cdot \\ \cdot \\ \frac{M_{21}}{\omega_{21}^2} \\ \frac{M_{22}}{\omega_{22}^2} \\ 0 \\ 0 \end{array} \right\} \left[ \begin{array}{cccccc} 0 & 1 & \psi_{01}(x_2) & \psi_{02}(x_2) & \psi_{03}(x_2) & 0 & \cdot & \cdot & \cdot & 0 \end{array} \right] \left[ \begin{array}{c} q_{0T} \\ \cdot \\ \cdot \\ \cdot \\ q_{21} \\ q_{22} \\ \alpha_2 \\ \mathcal{L}_1 \end{array} \right]
\end{array}$$

$$\begin{array}{c}
\left[ \begin{array}{cccccc} q_{0T} & \cdot & \cdot & \cdot & \alpha_1 & q_{21} & q_{22} & \alpha_2 & \mathcal{L}_1 \end{array} \right] \\
+ \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_2) \\ \psi_{02}(x_2) \\ \psi_{03}(x_2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{array} \right\} \left[ \begin{array}{cccccc} 0 & \cdot & \cdot & \cdot & 0 & \frac{M_{21}}{\omega_{21}^2} & \frac{M_{22}}{\omega_{22}^2} & 0 & 0 \end{array} \right] \left[ \begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ \mathcal{L}_1 \end{array} \right]
\end{array}$$

**Engine on second branch beam.** - For an engine on the branch beam, the stiffness

[illegible]

$$[B_{e2}] = m_{e2} \left\{ \begin{array}{c} 1 \\ x_{e2} + \epsilon_2 \\ \varphi_{01}(x_2) + (x_{e2} + \epsilon_2 - x_2)\psi_{01}(x_2) \\ \varphi_{02}(x_2) + (x_{e2} + \epsilon_2 - x_2)\psi_{02}(x_2) \\ \varphi_{03}(x_2) + (x_{e2} + \epsilon_2 - x_2)\psi_{03}(x_2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \varphi_{21}(x_{e2}) + \epsilon_2 \psi_{21}(x_{e2}) \\ \varphi_{22}(x_{e2}) + \epsilon_2 \psi_{22}(x_{e2}) \\ \epsilon_2 \\ 0 \end{array} \right\} \quad [\text{Transpose of column matrix}]$$

$$+ J_{e2} \left\{ \begin{array}{c} 0 \\ 1 \\ \psi_{01}(x_2) \\ \psi_{02}(x_2) \\ \psi_{03}(x_2) \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \psi_{21}(x_{e2}) \\ \psi_{22}(x_{e2}) \\ 1 \\ 0 \end{array} \right\} \quad \left[ \begin{array}{c} \text{Transpose of column matrix} \end{array} \right] \left\{ \begin{array}{c} q_{0T} \\ q_{0R} \\ q_{01} \\ q_{02} \\ q_{03} \\ \alpha_0 \\ \cdot \\ \cdot \\ \cdot \\ q_{21} \\ q_{22} \\ \alpha_2 \\ \varphi_1 \end{array} \right\}$$

## REFERENCES

1. Fowler, J.: Generalized Missile Dynamic Analysis. Part V - Calculation of Bending Modes. Rep. No. EM8-16, Space Technology Labs., Ramo-Wooldridge Corp., Apr. 7, 1958.
2. Beard, Nathan L.: A Flexibility Influence Coefficient Method for Determining the Mode Shapes and Natural Frequencies of Space Vehicles. Rep. No. MTP-AERO-63-58, NASA, Aug. 13, 1963.
3. Shaker, F.: A Method for Calculating the Normal Modes and Frequencies of a Branched Timoshenko Beam. NASA TN D-4560, 1968.
4. Gieseke, R. K.; Appleby, B.; and Tonelli, W.: General Missile Vibration Program No. 2342. Rep. No. GD/A-DDE64-050, General Dynamics/Astronautics, Aug. 20, 1964.
5. Scanlan, R. H.: An Analytical Study of the Landing Shock Effect of an Elastic Airplane. J. Aeron. Sci., vol. 15, no. 5, May 1948, pp. 300-304.
6. Hurty, Walter C.: Dynamic Analysis of Structural Systems by Component Mode Synthesis. Tech. Rep. 32-530 (NASA CR-53057), Jet Propulsion Lab., California Inst. Tech., Jan. 15, 1964.
7. Lukens, D. R.; Schmitt, A. F.; Broucek, G. T.: Approximate Transfer Functions for Flexible-Booster-and-Autopilot Analysis. Rep. No. AE61-0198, General Dynamics/Convair, Apr. 1961.
8. Hurty, Walter C.; and Rubinstein, M. F.: Dynamics of Structures. Prentice-Hall, Inc., 1964.
9. Faddeeva, V. N. (Curtis D. Benster, trans.): Computational Methods of Linear Algebra. Dover Publications, 1959.

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